Assignment # 6	Due on December 11th in class

1. Let X and Y be normed spaces, and let $T\colon X \longrightarrow Y$ be a linear operator. Prove that

$$\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| \ .$$

2. Let X be a vector space. For a subset D of X, we say that D is **convex**, if "$x, y \in D$ implies that $\lambda x + (1-\lambda)y \in D$, $\forall\, \lambda \in [0,1]$ ". We say that D is **balanced** if "$x \in D$ implies that $-x \in D$". We say that D is **absorbing** if "$\forall\, x \in D$, $\exists\, \lambda > 0$, such that $x \in kD$ for all $k > \lambda$ ".

For a subset D of X, assuming that D is convex, balanced and absorbing, we define the following **Minkowski functional** (with respect to D) as

$$\rho\colon X \longrightarrow \mathbb{R}_{\geq 0},\ x \mapsto \rho(x) = \inf\{\lambda \in \mathbb{R}_{\geq 0}\colon x \in \lambda D\} \ .$$

With these in mind, prove the following:

i) $\rho(-x) = \rho(x)$, $\forall\, x \in X$.

ii) $\rho(\lambda x) = |\lambda|\rho(x)$, $\forall\, x \in X$ and $\lambda \in \mathbb{R}$.

iii) $\rho(x+y) \leq \rho(x) + \rho(y)$, $\forall\, x, y \in X$.

<u>Hint:</u> To prove iii), you definitely need to use the fact that D is convex.